

Consequences of Classical Ignorance

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Generalized Quantum States from Classical Ignorance

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Imagine an imperfect device that yields $|\psi_1\rangle\langle\psi_1|$ with probability p_1 , and a different state $|\psi_2\rangle\langle\psi_2|$ with probability $p_2 = 1 - p_1$. To describe such a preparation we use a state operator of the form,

$$\hat{\rho} = p_1|\psi_1\rangle\langle\psi_1| + (1 - p_1)|\psi_2\rangle\langle\psi_2| \quad (1)$$

, where we are assuming $|\psi_1\rangle \in \mathcal{H}$, $|\psi_2\rangle \in \mathcal{H}$ such that $\langle\psi_1|\psi_2\rangle \neq 0$ and $\text{tr}(|\psi_{1,2}\rangle\langle\psi_{1,2}|) = 1$.

This state operator has the properties,

- $\text{tr}(\hat{\rho}) = 1$
- $\langle u|\hat{\rho}|u\rangle \geq 0$ for all $|u\rangle \in \mathcal{H}$
- $\text{tr}(\hat{\rho}^2) \leq 1$

If $\dim(\mathcal{H}) = d$, then

$$\frac{1}{d} \leq \text{tr}(\hat{\rho}^2) \leq 1. \quad (2)$$

The maximally mixed state $\hat{\rho} = \frac{\hat{1}}{d}$ represents total ignorance about the system.

The continuous variable generalization is

$$\hat{\rho} = \int d\alpha p(\alpha)|\psi(\alpha)\rangle\langle\psi(\alpha)|. \quad (3)$$

A mixed state can be ‘created’ in two ways: (i) as a convex combination of pure states due to classical ignorance (this is called a *proper mixture*), or (ii) by ignoring any quantum correlations between the system and some ancillary system, leaving only partial information about the system state (this is called an *improper mixture*).

Note that there is an infinite number of different convex combinations of pure states for the same mixed state (this is called *the ambiguity of mixtures*.)

Generalized Quantum Transformations from Classical Ignorance

Now consider a (non-ideal) device implementing with probability p_1 a unitary \hat{U}_1 , and with probability $1 - p_1$ implementing $\hat{U}_2 \neq \hat{U}_1$. A linear map $\mathcal{E}(\hat{\rho})$ defined by

$$\mathcal{E}(\hat{\rho}) = p_1\hat{U}_1\hat{\rho}\hat{U}_1^\dagger + (1 - p_1)\hat{U}_2\hat{\rho}\hat{U}_2^\dagger \equiv \hat{\rho}_f \quad (4)$$

is not unitary, but it is positive (i.e. it takes state operators to state operators).

In general, $\text{tr}(\mathcal{E}(\hat{\rho})^2) \leq \text{tr}(\hat{\rho}^2)$ with equality if $p_1 = 0$ or 1 , that is, unitary operators preserve the purity, but these linear operators can only decrease the purity.

The continuous variable generalization is given by

$$\mathcal{E}(\hat{\rho}) = \int d\alpha p(\alpha) \hat{U}(\alpha) \hat{\rho} \hat{U}(\alpha)^\dagger. \quad (5)$$

Generalized Quantum Measurements from Classical Ignorance

Take some Hermitian operator $\hat{O} = \hat{O}^\dagger$. Then

$$\hat{O} = \sum_k \lambda_k |\lambda_k\rangle \langle \lambda_k| \quad (6)$$

For some state $\hat{\rho}$, the probability of the outcomes are

$$\text{Pr}(k) = \text{tr}(\hat{\rho} |\lambda_k\rangle \langle \lambda_k|) = p_k \quad (7)$$

where the p_k can be estimated by the relative frequency of the number of outcomes k with respect to the total number of tests/experiments. With that information, one can calculate

$$\langle \hat{O} \rangle = \text{tr}(\hat{O} \hat{\rho}) = \sum_k \lambda_k p_k, \quad (8)$$

but also the expectation of any operator decomposed by the same projectors, e.g.

$$\langle \hat{O}^2 \rangle = \text{tr} \left(\sum_k \lambda_k^2 |\lambda_k\rangle \langle \lambda_k| \hat{\rho} \right) = \sum_k \lambda_k^2 p_k. \quad (9)$$

So the crucial thing here is not the Hermitian operator, but the projectors $\hat{P}_k = |\lambda_k\rangle \langle \lambda_k|$ over which it is decomposed, and their associated probabilities p_k . These projectors need not be rank one. In general, these measurements are called *projector valued measurements* (PVM), the term is due to von Neumann (1932).

Now consider a measurement device which implements a PVM $\{|\phi_i\rangle \langle \phi_i|\}$ with probability p_1 , and the PVM $\{|\psi_i\rangle \langle \psi_i|\}$ with probability $(1 - p_1)$, i.e.

$$\{\hat{E}_i\} = \{p_1 |\phi_i\rangle \langle \phi_i| + (1 - p_1) |\psi_i\rangle \langle \psi_i|\}, \quad (10)$$

which implies

$$p_i = p_1 \text{tr}(\hat{\rho} |\phi_i\rangle \langle \phi_i|) + (1 - p_1) \text{tr}(\hat{\rho} |\psi_i\rangle \langle \psi_i|) \quad (11)$$

$$= \text{tr}[\hat{\rho} (p_1 |\phi_i\rangle \langle \phi_i| + (1 - p_1) |\psi_i\rangle \langle \psi_i|)] \quad (12)$$

$$= \text{tr}[\hat{\rho} \hat{E}_i] \quad (13)$$

and

$$\langle u | \hat{E}_i | u \rangle \geq 0 \text{ for all } |u\rangle \in \mathcal{H} \quad (14)$$

so the \hat{E}_i are positive operators. Also, $\sum_i p_i = 1$ iff $\sum_i \hat{E}_i = 1$.

The \hat{E}_i are called *positive operator valued measurements* (POVM). Note that in general $\hat{E}_i^2 \neq \hat{E}_i$, and that PVMs are a special case where $\hat{E}_i^2 = \hat{E}_i$.

Any operator \hat{O} has a matrix representation in a basis $\{|\phi_i\rangle\}$. The matrix elements are given by $\langle \phi_i | \hat{O} | \phi_j \rangle = O_{ij}$ where i is the row number, and j is the column number.

Tensor Product Structure and Composite Systems

Say we have two systems S_A and S_B , and the total system $S = S_A + S_B$. The Hilbert space of the total system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and we use the convention that $|\psi_A\rangle \in \mathcal{H}_A$, $|\psi_B\rangle \in \mathcal{H}_B$ and $|\psi\rangle \in \mathcal{H}$.

Let $\{|\psi_{A(B),i}\rangle\}$ be a basis for $\mathcal{H}_{A(B)}$, so that, for example, $|\psi_A\rangle = \sum a_i |\psi_{A,i}\rangle$, where $a_i = \langle \phi_{A,i} | \psi_A \rangle$ and $|\psi\rangle = \sum_{ij} c_{i,j} |\phi_{A,i}\rangle \otimes |\phi_{B,j}\rangle$ so the $\{|\phi_{A,i}\rangle |\phi_{B,j}\rangle\}$ form a basis for \mathcal{H} .

An operator $\hat{A} \otimes \hat{B}$ acts as $\hat{A} \otimes \hat{B} |\psi\rangle = \sum_{ij} c_{ij} \hat{A} |\phi_{A,i}\rangle \otimes \hat{B} |\phi_{B,j}\rangle$.

Any operator \hat{O} acting on \mathcal{H} can be expressed $\hat{O} = \sum_k \hat{A}_k \otimes \hat{B}_k$.

Define $\hat{O} \equiv \hat{A} \otimes \hat{1}$ then $\langle \hat{O} \rangle = \text{tr}[(\hat{A} \otimes \hat{1}) |\psi\rangle \langle \psi|]$ We can define $\hat{\rho}_A = \text{tr}_B |\psi\rangle \langle \psi|$ so that $\langle \hat{O} \rangle = \text{tr}[\hat{\rho}_A \hat{A}]$.

How is the partial trace calculated exactly?

$$\hat{\rho}_A = \text{tr}_B(|\psi\rangle \langle \psi|) \quad (15)$$

$$= \sum_j \langle \phi_{B,j} | \psi \rangle \langle \psi | \phi_{B,j} \rangle \quad (16)$$

which uniquely defines a state operator over subsystem S_A .

Entanglement

A pure state $\rho = |\psi\rangle \langle \psi|$, where $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled (not factorable, not separable) iff it cannot be expressed as

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \quad (17)$$

There is a simple criterion for entanglement in pure states: $\text{tr}(\hat{\rho}_A^2) < 1$. That is, if the reduced density matrix is pure, then the system is not entangled.

Another useful result is the Schmidt decomposition. For all $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there are bases $\{|\chi_{A,i}\rangle\} \{|\chi_{B,i}\rangle\}$ for \mathcal{H}_A and \mathcal{H}_B respectively such that

$$|\psi\rangle = \sum_i \alpha_i |\chi_{A,i}\rangle \otimes |\chi_{B,i}\rangle \quad (18)$$

where $\alpha_i \geq 0$, $\sum_i \alpha_i^2 = 1$. $|\psi\rangle$ is entangled iff the number of non-zero α_i is greater than 1. Using the Schmidt decomposition it easy to show that

$$\text{tr}(\hat{\rho}_A^2) = \text{tr}(\hat{\rho}_B^2) \text{ if } |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B. \quad (19)$$

Generalized Postulates of Quantum Theory

Similar to the way we can obtain a mixed state on \mathcal{H}_A from a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ by ignoring one of the subsystems (tracing over it), we can also obtain generalized transformations $\mathcal{E}(\hat{\rho}_{i,A}) = \hat{\rho}_{f,A}$ on \mathcal{H}_A that are not unitary by tracing over the B part of a unitary operator \hat{U} acting on $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. We can also obtain POVMs on \mathcal{H}_A by tracing out the B component of a PVM acting on $\mathcal{H}_A \otimes \mathcal{H}_B$.

Postulate I. The most general description of a system's configuration is given by a non-negative, unit trace, state operator $\hat{\rho}$. Rank-one projectors $\hat{\rho} = |\psi\rangle \langle \psi|$ are state of maximal knowledge.

Postulate II. Every measurement can be described by a set $\{\hat{E}_k\}$ of positive operators such that $\sum_k \hat{E}_k = 1$. The probability of the outcome labeled by k , for state $\hat{\rho}$, is given by $\text{Pr}(k) = p_k = \text{tr}(\hat{\rho}\hat{E}_k)$. Rank one PVMs correspond to maximal tests.

Postulate III. The most general description of a transformation is a completely positive linear map $\mathcal{E}(\hat{\rho})$ which takes the set of positive operators to itself. A *completely positive* (CP) operator is an operator \mathcal{E} such that $\mathcal{E}(\hat{\rho}_A \otimes \hat{1}) = \hat{\rho}_{AB}$ where both $\hat{\rho}_A \otimes \hat{1}$ and $\hat{\rho}_{AB}$ are both non-negative. Physically realizable operations are required to be CP – for example, the partial transpose is not a CP map.

While any mixed state can be represented by classical ignorance, there are general operations that cannot be represented by classical ignorance over unitary operations. Some general operations can only arise when an auxiliary quantum system has been traced over.

Sequential Measurements: von Neumann on the Compton Experiment

In the experiment where photons are scattered off electrons that are initially at rest (by Compton), von Neumann considered the scenario where there is a finite time difference Δt between the interception of e^- and the photon (the detection of these particles). There are 3 logically possible degrees of randomness:

- 1) The momentum of the e^- and the photon have uncorrelated dispersions
- 2) The momentum of e^- has a dispersion, but that momentum of the photon is in a fixed correlation with the momentum of the e^-
- 3) Both particles' momenta can be predicted (from the initial conditions) with certainty (no dispersion)

Experiments tell us that (2) is the actual case. This is the practical reason why von Neumann was forced to introduce the collapse postulate.

Projection Postulate (von Neumann, 1932)

von Neumann realized from the analysis of sequential measurements that two kinds of transformation were required in quantum mechanics:

Process 1. After observation/measurement of an outcome λ_k , the system is left in the eigenstate $|\lambda_k\rangle$ associated with the detected eigenvalue λ_k . We have the map,

$$\hat{\rho} \rightarrow |\lambda_k\rangle\langle\lambda_k|.$$

von Neumann called this process an essential randomness in nature, and he considered it grounds for abandoning the “principle of sufficient cause.”

Process 2. The normal Schrödinger (unitary) evolution.